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A uniqueness theorem for meromorphic functions ignoring multiplicity

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Abstract. In this paper, we give a uniqueness theorem for meromorphic functions ignoring multiplicity , which generalizes a An's theorem in [1].

1. Introduction. Main results

In this paper, by a meromorphic function we mean a meromorphic function on the complex plane $\mathbb{C}$.

Let f be a non-constant meromorphic function on $\mathbb{C}$. For every $a \in \mathbb{C}$, we define the function $\nu_f^a : \mathbb{C} \to \mathbb{N}$ by

$$\nu_f^a(z) = \begin{cases} 0 & \text{if } f(z) \neq a \\ d & \text{if } f(z) = a \text{ with multiplicity } d, \end{cases}$$

and set $\nu_f^\infty = \nu_{\frac{1}{f}}^0$, and define the function $\overline{\nu}_f^a : \mathbb{C} \to \mathbb{N}$ by $\overline{\nu}_f^a(z) = \min \{\nu_f^a(z), 1\}$, and set $\overline{\nu}_f^\infty = \overline{\nu}_{\frac{1}{f}}^0$. For $f \in \mathcal{M}(\mathbb{C})$ and a non-empty set $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E_f(S) = \bigcup_{a \in S} \{(z, \nu_f^a(z)) : z \in \mathbb{C}\}, \quad \overline{E}_f(S) = \bigcup_{a \in S} \{(z, \overline{\nu}_f^a(z)) : z \in \mathbb{C}\}.$$

Let $\mathcal{F}$ be a nonempty subset of $\mathcal{M}(\mathbb{C})$. Two functions f, g of $\mathcal{F}$ are said to *share* S, *counting multiplicity* (share S CM) if $E_f(S) = E_g(S)$, and to *share* S, *ignoring multiplicity* (share S IM) if $\overline{E}_f(S) = \overline{E}_g(S)$.

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If the condition $E_f(S) = E_g(S)$ implies $f = g$ for any two non-constant meromorphic (entire) functions f, g, then S is called a unique range set for meromorphic (entire) functions counting multiplicity, or in brief, URSM (URSE). A set $S \subset \mathbb{C} \cup \{\infty\}$ is called a unique range set for meromorphic (entire) functions ignoring multiplicity, or in brief, URSM-IM (URSE-IM), if the condition $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f = g$ for any pair of non-constant meromorphic (entire) functions.

In 1976 Gross ([10]) proved that there exist three finite sets S_j $(j = 1, 2, 3)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$, $j = 1, 2, 3$ must be identical. In the same paper Gross([10]) posed the following question:

Question A. *Can one find two (or possible even one) finite set S_j $(j = 1, 2)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ $(j = 1, 2)$ must be identical?*

Yi ([18]-[20],[22]) first gave an affirmative answer to Question A. Since then, many results have been obtained for this and related topics (see ([1]-[15]), ([17]-[23])).

Concerning to Question A, a natural question is the following.

Question B. *What is the smallest cardinality for such a finite set S such that any two meromorphic functions f and g satisfying either $E_f(S) = E_g(S)$ or $\overline{E}_f(S) = \overline{E}_g(S)$ must be identical?*

So far, the best answer to Question B for the case of URSM was obtained by Frank and Reinders ([7]). They proved the following result.

Theorem C. *The set $\{z \in \mathbb{C} |\ P_{FR}(z) = \frac{(n-1)(n-2)}{2} z^n + n(n-2)z^{n-1} + \frac{(n-1)n}{2} z^{n-2} - c = 0\}$, where $n \geq 11$ and $c \neq 0, 1$, is a unique range set for meromorphic functions counting multiplicity.*

In 1997, H. X. Yi ([21]) first gave an answer to question B for the case of URSM-IM with 19 elements. Since then, many results have been obtained for this topic (see ([1]- [5])). So far, the best answer to Question B for the case of URSM-IM was obtained by Chakraborty([5]). He proved the following result.

Theorem D. *Let $S_{FR} = \{z \in \mathbb{C} |\ P_{FR}(z) = 0\}$. If $n \geq 15$, then S_{FR} is a URSM-IM.*

In 2022, An([1]) given a class of unique range sets for meromorphic functions ignoring multiplicity with 15 elements. He proved the following result.

Let $n \in \mathbb{N}^*, n \geq 3$. Consider polynomial $P(z)$ as follows:

$$P_A(z) = z^n - \frac{2na}{n-1} z^{n-1} + \frac{na^2}{n-2} z^{n-2} + 1 = Q_A(z) + 1, \tag{1.1}$$

where $a \in \mathbb{C},\ a \neq 0$. Suppose that

$$Q_A(a) \neq -1, \tag{1.2}$$

$$Q_A(a) \neq -2. \tag{1.3}$$

Theorem E. *Let $P_A(z)$ be defined by (1.1) with conditions (1.2) and (1.3), and let $S_A = \{z \in \mathbb{C} |\ P_A(z) = 0\}$. If $n \geq 15$, then S_A is a URSM-IM.*

Clearly, $P_A^{'}(z) = nz^{n-3}(z-a)^2$, and $P_{FR}^{'}(z) = \frac{n(n-1)(n-2)}{2} z^{n-3}(z-1)^2$. Therefore, this class is different from Chakraborty's Theorem D in([5]).

In this paper, we give a uniqueness theorem for meromorphic functions ignoring multiplicity, which generalizes Theorem E.

Now let us describe main results of the paper.

Let $q, k, m_1, m_2 \in \mathbb{N}^*$.

We will let $P(z)$ be polynomial having no multiple zeros of degree q in $\mathbb{C}[z]$:

$$P(z) = (m_1+m_1+1)\Big(\sum_{i=0}^{m_2} \binom{m_2}{i} \frac{(-1)^i}{m_1+m_2+1-i} z^{m_1+m_2+1-i} a^i\Big) + 1 = Q(z)+1,$$

where

$$Q(z) = (m_1+m_2+1)\Big(\sum_{i=0}^{m_2} \binom{m_2}{i} \frac{(-1)^i}{m_1+m_2+1-i} z^{m_1+m_2+1-i} a^i\Big). \tag{1.4}$$

Suppose that

$$a \neq 0,\ Q(a) \neq -1,\ Q(a) \neq -2. \tag{1.5}$$

Clearly, $P^{'}(z) = (m_1+m_2+1)z^{m_1}(z-a)^{m_2}$, and has a zero at 0 of order m_1, and a zero at a of order m_2. Note that $q = m_1 + m_2 + 1$.

We shall prove the following theorem.

Theorem 1. *Let $P(z)$ be defined in (1.4) with conditions (1.5), and let $S = \{z \in \mathbb{C} |\ P(z) = 0\}$. If $q \geq 15$, then S is a URSM-IM.*

Remark 2. *From proof of Theorem 1 we give a proof of Theorem D, which is different from Chakraborty's proof in([5])*(see section 3.).

Remark 3. *In Theorem 1, take $m_1 = n-3$ and $m_2 = 2$ we obtain Theorem E.*

Indeed, by $P_A^{'}(z) = nz^{n-3}(z-a)^2$ and $P^{'}(z) = (m_1+m_2+1)z^{m_1}(z-a)^{m_2}$, we obtain $P(z) = P_A(z)$ when $m_1 = n-3$, $m_2 = 2$.

2. Lemmas, Definitions

We assume that the reader is familiar with the notations of Nevanlinna theory (see, for example, ([6]), ([16])). We need some lemmas.

Lemma 2.1. ([6], paper 98;[16], paper 43) *Let f be a non-constant meromorphic function on $\mathbb{C}$ and let $a_1, a_2, ..., a_q$ be distinct points of $\mathbb{C} \cup \{\infty\}$. Then*

$$(q-2)T(r,f) \leq \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) - N_0(r, \frac{1}{f'}) + S(r,f),$$

where $N_0(r, \frac{1}{f'})$ is the counting function of those zeros of f', which are not zeros of function $(f-a_1)...(f-a_q)$, and $S(r,f) = o(T(r,f))$ for all r, except for a set of finite Lebesgue measure.

Lemma 2.2. ([6, paper 99]) *For any non-constant meromorphic function f,*

$$T(r, \frac{1}{f'}) \leq 2T(r,f) + S(r,f).$$

Definition. Let f be a non-constant meromorphic function, and k be a positive integer. We denote by $\overline{N}_{(k}(r,f)$ the counting function of the poles of order $\geq k$ of f, where each pole is counted only once. If z is a zero of f, denote by $\nu_f(z)$ its multiplicity. We denote by $\overline{N}(r, \frac{1}{f'}; f \neq 0)$ the counting function of the zeros z of $f^{'}$ satisfying $f(z) \neq 0$, where each zero is counted only once.

Let be given two non-constant meromorphic functions f and g. For simplicity, denote by $\nu_1(z) = \nu_f(z)$ (resp.,$\nu_2(z) = \nu_g(z)$), if z is a zero of f(resp.,g). Let $f^{-1}(0) = g^{-1}(0)$. We denote by $N(r, \frac{1}{f}; \nu_1 = \nu_2 = 1)$(resp., $\overline{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1)$) the counting function of the common zeros z, satisfying $\nu_1(z) = \nu_2(z) = 1$(resp., $\nu_1(z) > \nu_2(z) \geq 1$, where each zero is counted only once), and by $N(r, \frac{1}{f}; \nu_1 \geq 2)$ the counting function of the zeros z of f, satisfying $\nu_1(z) \geq 2$. Similarly, we define the counting functions $\overline{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1)$, $N(r, \frac{1}{g}; \nu_2 \geq 2)$.

Lemma 2.3. ([1, Lemma 2.3])

Let f, g be two non-constant meromorphic functions and let $f^{-1}(0) = g^{-1}(0)$. Set

$$F = \frac{1}{f},\ G = \frac{1}{g},\ L = \frac{F^{''}}{F'} - \frac{G^{''}}{G'}.$$

Suppose that $L \not\equiv 0$. *Then*

$$1)\ N(r,L) \le \overline{N}_{(2}(r,f) + \overline{N}_{(2}(r,g) + \overline{N}(r,\frac{1}{f};\nu_1 > \nu_2 \ge 1) + \overline{N}(r,\frac{1}{g};\nu_2 > \nu_1 \ge 1) + \overline{N}(r,\frac{1}{f'}; f \ne 0) + \overline{N}(r,\frac{1}{g'}; g \ne 0).$$

Moreover, if a *is a common simple zero of* f *and* g, *then* $L(a) = 0$.

$$2)\ \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{f};\nu_1 > \nu_2 \ge 1) + \overline{N}(r,\frac{1}{g};\nu_2 > \nu_1 \ge 1) \le N(r,L) + \frac{1}{2}(N(r,\frac{1}{f}) + N(r,\frac{1}{g})) + N(r,\frac{1}{f};\nu_1 \ge 2) + N(r,\frac{1}{g};\nu_2 \ge 2) + S(r,f) + S(r,g).$$

A polynomial $R(z)$ is called a *strong uniqueness polynomial for meromorphic (entire) functions* if for arbitrary two non-constant meromorphic (entire) functions f and g, and a nonzero constant c, the condition $R(f) = cR(g)$ implies $f = g$ (see ([2]), ([9]), ([13])). In this case we say $R(z)$ is a SUPM (SUPE). A polynomial $R(z)$ is called a *uniqueness polynomial for meromorphic (entire) functions* if for arbitrary two non-constant meromorphic (entire) functions f and g, the condition $R(f) = R(g)$ implies $f = g$ (see ([2]), ([9]), ([13])). In this case we say $R(z)$ is a UPM (UPE). Let $R(z)$ be a polynomial of the degree q. Assume that the derivative of $R(z)$ has mutually distinct k zeros $d_1, d_2, ..., d_k$ with multiplicities $q_1, q_2, ..., q_k$, respectively. We often consider polynomials satisfying the following condition introduced by Fujimoto ([8]):

$$R(d_i) \ne R(d_j), 1 \le i < j \le q. \tag{2.1}$$

The number k is called the *derivative index* of R.

H. Fujimoto ([8], Proposition 7.1)) proved the following:

Lemma 2.4. *Let* $R(z)$ *be a polynomial of degree* q *satisfying the condition (2.1), we assume furthermore that* $q \ge 5$ *and there are two non-constant meromorphic function* f *and* g *such that*

$$\frac{1}{R(f)} = \frac{c_0}{R(g)} + c_1$$

for two constants $c_0 \ne 0$ *and* c_1. *If* $k \ge 3$ *or if* $k = 2, min\{q_1, q_2\} \ge 2$, *then* $c_1 = 0$.

Lemma 2.5. ([13], Theorem 1.1)

Let $P(z)$ be defined by (1.4) with conditions (1.5), and let $n \geq 6$. Then $P(z)$ is a strong uniqueness polynomial for meromorphic functions.

Lemma 2.6. ([3], Theorem 1.1)

Let $P_{FR_1}(z) = \frac{(n-1)(n-2)}{2}z^n + n(n-2)z^{n-1} + \frac{(n-1)n}{2}z^{n-2} - c = 0\}$, where $n \geq 8$ and $c \in \mathbb{C}$. Then $P_{FR_1}(z)$ is a strong uniqueness polynomial for meromorphic functions.

3. Proof of Theorems

Proof of Theorem 1

Recall that $P(z) = (z - a_1)...(z - a_q)$, $P^{'}(z) = qz^{m_1}(z - a)^{m_2}$, $q = m_1 + m_2 + 1$.

Suppose $q \geq 15$ and $\overline{E}_f(S) = \overline{E}_g(S)$, where $S = \{z \in \mathbb{C}|\ P(z) = 0\}$. Set

$$F = \frac{1}{P(f)},\ G = \frac{1}{P(g)}, L = \frac{F^{''}}{F^{'}} - \frac{G^{''}}{G^{'}},$$

$$T(r) = T(r, f) + T(r, g), S(r) = S(r, f) + S(r, g).$$

Then $T(r, P(f)) = qT(r, f) + S(r, f)$ and $T(r, P(g)) = qT(r, g) + S(r, g)$, and hence $S(r, P(f)) = S(r, f)$ and $S(r, P(g)) = S(r, g)$.

We consider two following cases:

Case 1. $L \equiv 0$. Then, we have $\frac{1}{P(f)} = \frac{c}{P(g)} + c_1$ for some constants $c \neq 0$ and c_1. By Lemma 2.4 we obtain $c_1 = 0$.

Therefore, there is a constant $C \neq 0$ such that $P(f) = CP(g)$. Then, applying Lemma 2.5 we obtain $f = g$.

Case 2. $L \not\equiv 0$.

Claim 1. We have

$$(q - 2)T(r) \leq \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) - N_0(r, \frac{1}{f^{'}}) - N_0(r, \frac{1}{g^{'}}) + S(r), \quad (3.1)$$

where $N_0(r, \frac{1}{f^{'}})$ ($N_0(r, \frac{1}{g^{'}})$) is the counting function of those zeros of f', which are not zeros of function $(f - a_1)...(f - a_q)f(f - a)((g - a_1)...(g - a_q)g(g - a))$.

Indeed, applying the Lemma 2.1 to the functions f, g and the values $a_1, a_2, ...,$ a_q, $0, a, \infty$, and noting that

$$\sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) = \overline{N}(r, \frac{1}{P(f)}),\ \sum_{i=1}^{q} \overline{N}(r, \frac{1}{g-a_i}) = \overline{N}(r, \frac{1}{P(g)}),$$

we obtain

$$(q+1)T(r) \leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) +$$

$$\overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{g-a}) - N_0(r, \frac{1}{f'}) - N_0(r, \frac{1}{g'}) + S(r). \tag{3.2}$$

On the other hand,

$$\overline{N}(r, f) + \overline{N}(r, g) \leq (T(r, f) + T(r, g)) + S(r) = T(r) + S(r),$$

$$\overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) \leq (T(r, f) + T(r, g)) + S(r) = T(r) + S(r),$$

$$\overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{g-a}) \leq (T(r, f) + T(r, g)) + S(r) = T(r) + S(r).$$

From this and (3.2) we obtain (3.1).

Claim 2. We have

$$\overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) \leq$$

$$(\frac{q}{2} + 3)T(r) + \overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) + S(r).$$

Indeed, by $\overline{E}_f(S) = \overline{E}_g(S)$ we get $(P(f))^{-1}(0) = (P(g))^{-1}(0)$. For simplicity, we set $\nu_1 = \nu_1(z)$, $\nu_2 = \nu_2(z)$, where $\nu_1(z) = \nu_{P(f)}(z)$, $\nu_2(z) = \nu_{P(g)}(z)$. Note that

$$\overline{N}_{(2}(r, P(f)) = \overline{N}(r, f),\ \overline{N}_{(2}(r, P(g)) = \overline{N}(r, g),$$

$$S(r, P(f)) = S(r, f),\ S(r, P(g)) = S(r, g), S(r) = S(r, f) + S(r, g).$$

Applying the Lemma 2.3 to the functions $P(f), P(g)$. Then we obtain

$$N(r, L) \leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}(r, \frac{1}{P(f)}; \nu_1 > \nu_2 \geq 1) + \overline{N}(r, \frac{1}{P(g)}; \nu_2 > \nu_1 \geq 1)$$

$$+\overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0), \tag{3.3}$$

and

$$\overline{N}(r,\frac{1}{P(f)})+\overline{N}(r,\frac{1}{P(g)})+\overline{N}(r,\frac{1}{P(f)};\nu_1>\nu_2\geq 1)+$$

$$\overline{N}(r,\frac{1}{P(g)};\nu_2>\nu_1\geq 1)\leq N(r,L)+\frac{1}{2}(N(r,\frac{1}{P(f)})+N(r,\frac{1}{P(g)}))+$$

$$N(r,\frac{1}{P(f)};\nu_1\geq 2)+N(r,\frac{1}{P(g)};\nu_2\geq 2))+S(r). \tag{3.4}$$

Morover,

$$\overline{N}(r,f)+\overline{N}(r,g)\leq T(r)+S(r). \tag{3.5}$$

Obviously,

$$N(r,\frac{1}{P(f)})\leq qT(r,f)+S(r,f); N(r,\frac{1}{P(g)})\leq qT(r,g)+S(r,g),$$

$$N(r,\frac{1}{P(f)})+N(r,\frac{1}{P(g)})\leq qT(r)+S(r). \tag{3.6}$$

On the other hand, from $P(f)=(f-a_1)...(f-a_q)$ it follows that if z_0 zero is a zero of $P(f)$ with multiplicity ≥ 2, then z_0 is a zero of $f-a_i$ with multiplicity ≥ 2 for some $i\in\{1,2,...,q\}$, and therefore, it is a zero of $f^{'}$, so we have

$$N(r,\frac{1}{P(f)};\nu_1\geq 2)\leq N(r,\frac{1}{f'}).$$

From this and Lemma 2.2 we obtain

$$N(r,\frac{1}{P(f)};\nu_1\geq 2)\leq N(r,\frac{1}{f'})\leq T(r,f^{'})+S(r,f)\leq 2T(r,f)+S(r,f).$$

Similarly, we have

$$N(r,\frac{1}{P(g)};\nu_2\geq 2)\leq N(r,\frac{1}{g'})\leq T(r,g^{'})+S(r,g)\leq 2T(r,g)+S(r,g).$$

Therefore,

$$N(r,\frac{1}{P(f)};\nu_1\geq 2)+N(r,\frac{1}{P(g)};\nu_2\geq 2)\leq 2T(r)+S(r). \tag{3.7}$$

Combining (3.1)-(3.7) we get

$$\overline{N}(r,\frac{1}{P(f)})+\overline{N}(r,\frac{1}{P(g)})\leq$$

$$(\frac{q}{2}+3)T(r)+\overline{N}(r,\frac{1}{[P(f)]'};P(f)\neq 0)+\overline{N}(r,\frac{1}{[P(g)]'};P(g)\neq 0)+S(r).$$

Claim 2 is proved.

Claim 3. We have

$$\overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) \leq 2T(r) + N_0(r, \frac{1}{f'}) + N_0(r, \frac{1}{g'}) + S(r).$$

We have

$$\overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) = \overline{N}(r, \frac{1}{f^{m_1}(f-a)^{m_2} f'}; P(f) \neq 0) \leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-a}) + \overline{N}_0(r, \frac{1}{f'}) \leq 2T(r,f) + \overline{N}_0(r, \frac{1}{f'}) + S(r,f). \tag{3.8}$$

Similarly,

$$\overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) \leq 2T(r,g) + \overline{N}_0(r, \frac{1}{g'}) + S(r,g). \tag{3.9}$$

Inequalities (3.8) and (3.9) give us

$$\overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) \leq \leq 2T(r) + \overline{N}_0(r, \frac{1}{f'}) + \overline{N}_0(r, \frac{1}{g'}) + S(r).$$

Claim 3 is proved.

Claim 1, 2, 3 give us:

$$(q-2)T(r) \leq (\frac{q}{2} + 5)T(r) + S(r). \text{ So } (q-14)T(r) \leq S(r).$$

This is a contradiction to the assumption that $q \geq 15$. So $L \equiv 0$. Therefore $f = g$. Theorem 1 is proved.

A proof of Theorem D

By using the arguments similar in proof of Theorem 1 and Lemma 2.6 we give a proof of Theorem D, which is different from Chakraborty's proof of Theorem D in([5]).

Recall that $P_{FR}(z) = (z-a_1)...(z-a_n)$, $P'_{FR}(z) = \frac{n(n-1)(n-2)}{2} z^{n-3}(z-1)^2$.

Suppose $n \geq 15$ and $\overline{E}_f(S_{FR}) = \overline{E}_g(S_{FR})$, where $S_{FR} = \{z \in \mathbb{C} |\ P_{FR}(z) = 0\}$. Set

$$F = \frac{1}{P_{FR}(f)},\ G = \frac{1}{P_{FR}(g)}, L = \frac{F^{''}}{F'} - \frac{G^{''}}{G'},$$
$$T(r) = T(r, f) + T(r, g), S(r) = S(r, f) + S(r, g).$$

Then $T(r, P_{FR}(f)) = nT(r, f) + S(r, f)$ and $T(r, P_{FR}(g)) = nT(r, g) + S(r, g)$, and hence $S(r, P_{FR}(f)) = S(r, f)$ and $S(r, P_{FR}(g)) = S(r, g)$.

We consider two following cases:

Case 1. $L \equiv 0$. Then, we have $\frac{1}{P_{FR}(f)} = \frac{c}{P_{FR}(g)} + c_1$ for some constants $c \neq 0$ and c_1. By Lemma 2.4 we obtain $c_1 = 0$.

Therefore, there is a constant $C \neq 0$ such that $P_{FR}(f) = CP_{FR}(g)$. Then, applying Lemma 2.6 we obtain $f = g$.

Case 2. $L \not\equiv 0$. By using the arguments similar in proof of Theorem 1 we obtain

Claim 1. We have

$$(n-2)T(r) \leq \overline{N}(r, \frac{1}{P_{FR}(f)}) + \overline{N}(r, \frac{1}{P_{FR}(g)}) - N_0(r, \frac{1}{f'}) - N_0(r, \frac{1}{g'}) + S(r), \tag{3.10}$$

where $N_0(r, \frac{1}{f'})$ ($N_0(r, \frac{1}{g'})$) is the counting function of those zeros of f', which are not zeros of function $(f-a_1)...(f-a_n)f(f-1)((g-a_1)...(g-a_n)g(g-1))$.

Claim 2. We have

$$\overline{N}(r, \frac{1}{P_{FR}(f)}) + \overline{N}(r, \frac{1}{P_{FR}(g)}) \leq$$
$$(\frac{n}{2}+3)T(r)+\overline{N}(r, \frac{1}{[P_{FR}(f)]'}; P_{FR}(f) \neq 0)+\overline{N}(r, \frac{1}{[P_{FR}(g)]'}; P_{FR}(g) \neq 0)+S(r).$$

Claim 3. We have

$$\overline{N}(r, \frac{1}{[P_{FR}(f)]'}; P_{FR}(f) \neq 0)+\overline{N}(r, \frac{1}{[P_{FR}(g)]'}; P_{FR}(g) \neq 0) \leq 2T(r)+N_0(r, \frac{1}{f'})+$$
$$N_0(r, \frac{1}{g'}) + S(r).$$

Claim 1, 2, 3 give us:

$$(n-2)T(r) \leq (\frac{n}{2} + 5)T(r) + S(r). \text{ So } (n-14)T(r) \leq S(r).$$

This is a contradiction to the assumption that $n \geq 15$. So $L \equiv 0$. Therefore $f = g$.

Theorem D is proved.

References

[1] Vu Hoai An, *A new class of unique range sets for meromorphic functions ignoring multiplicity with* 15 *elements,* Journal of Mathematics and Mathematical Sciences(2022), Vol. 1, 1-14.

[2] A. Banerjee, *A new class of strong uniqueness polynomial satisfying Fujimoto's conditions,* Ann. Acad. Sci. Fenn. Math. Vol. 40, 2015, 465-474.

[3] A. Banerjee, B. Chakraborty, S. Mallickc, *Further Investigations on Fujimoto Type Strong Uniqueness Polynomials*, Filomat 31:16 (2017), 5203-5216.

[4] S. Bartels, *Meromorphic functions sharing a set with 17 elements ignoring multiplicities,* Compl. Var. Theory Appl., 39, 85-92 (1999).

[5] B. Chakraborty, *On the Cardinality of a Reduced Unique-Range Set*, Ukr. Math. J., Vol. 72, No. 11, April, 2021, DOI 10.1007/s11253-021-01889-z.

[6] A. A. Goldberg and I. V. Ostrovskii, *Value Distribution of Meromorphic Functions*, Translations of Mathematical Monographs (2008), V.236.

[7] G.Frank and M. Reinders, *A unique range set for meromorphic functions with 11 elements,* Compl. Var. Theory Appl. 37:1, 1998, 185-193.

[8] H. Fujimoto, *On uniqueness of meromorphic functions sharing finite sets,* Amer. J. Math. 122, 2000, 1175-1203.

[9] H.Fujimoto, *On uniqueness polynomials for meromorphic functions,* Nagoya Math. J., 170, 33-46 (2003).

[10] F.Gross, *Factorization of meromorphic functions and some open problems*, Complex Analysis(Proc. Conf . Univ. Kentucky, Lexington, Ky. 1976), pp. 51-69, Lecture Notes in Math. Vol. 599, Springer, Berlin, 1977.

[11] Ha Huy Khoai, Some remarks on the genericity of unique range sets for meromorphic functions, Sci. China Ser. A Mathematics, Vol. 48, 2005, 262-267.

[12] Ha Huy Khoai, Vu Hoai An, and Pham Ngoc Hoa,*On functional equations for meromorphic functions and applications,* Arch. Math, DOI 10.1007/s00013- 017-1093-5, 2017.

[13] Ha Huy Khoai, Vu Hoai An and Nguyen Xuan Lai, *Strong uniqueness polynomials of degree 6 and unique range sets for powers of meromorphic functions,* Intern. J. Math., 2018, DOI:10.1142/S0129167X18500374.

[14] Ha Huy Khoai, Vu Hoai An and Le Quang Ninh,*Value-sharing and uniqueness for L-functions,* Ann. Polonici Math.,2021, 265-278.

[15] Ha Huy Khoai and Vu Hoai An,*Determining an L-function in the extended Selberg class by its preimages of subsets,* Ramanujan Journal, 58, 253-267 (2022).

[16] W.K.Hayman, *Meromorphic Functions*, Clarendon, Oxford(1964).

[17] P.Li and C.C. Yang, *Some further results on the unique range sets of meromorphic functions,* Kodai Math. J. 18, 1995, 437-450.

[18] H.X.Yi , *Uniqueness of Meromorphic Functions and question of Gross*, Sci. China (Ser. A), Vol.37 N0.7, July 1994, 802-813.

[19] H. X. Yi, *A question of Gross and the uniqueness of entire functions*, Nagoya Math. J. Vol. 138 (1995), 169-177.

[20] H. X. Yi, *Unicity theorems for meromorphic and entire functions III,* Bull. Austral. Math. Soc., 53, 71–82 (1996).

[21] H. X. Yi,*The reduced unique range sets for entire or meromorphic functions,* Compl. Var. Theory Appl., 32, 191–198 (1997).

[22] H. X. Yi, *On a question of Gross concerning uniqueness of entire functions*, Bull. Austral. Math. Soc. Vol. 57(1998), 343-349.

[23] H. X. Yi and W.C.Lin, *Uniqueness theorems concerning a question of Gross*, Proc. Japan Acad., Ser. A, 80, 2004, 136-140.

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