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Truncated second main theorem for non-Archimedean meromorphic maps

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Abstract. Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \geq 0$, which is complete with respect to a non-Archimedean absolute value. Let V be a projective subvariety of $\mathbb{P}^M(\mathbb{F})$. In this paper, we will prove some second main theorems for non-Archimedean meromorphic maps of $\mathbb{F}^m$ into V intersecting a family of hypersurfaces in $N-$subgeneral position with truncated counting functions.

1. Introduction and Main results

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \geq 0$, which is complete with respect to a non-Archimedean absolute value. Let $N \geq n$ and $q \geq N+1$. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{F})$. The family of hyperplanes $\{H_1\}_{i=1}^q$ is said to be in N-subgeneral position in $\mathbb{P}^n(\mathbb{F})$ if $H_{j_0} \cap \cdots \cap H_{j_N} = \varnothing$ for every $1 \leq j_0 < \cdots < j_N \leq q$.

In 2017, Yan [6] proved a truncated second main theorem for a non - Archimedean meromorphic map into $\mathbb{P}^n(\mathbb{F})$ with a family of hyperplanes in subgeneral position. With the standart notations on the Nevanlinna theory for non-Archimedean meromorphic maps, his result is stated as follows.

Theorem A (cf. [6, Theorem 4.6]) *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \geq 0$, which is complete with respect to a non-Archimedean absolute value. Let $f : \mathbb{F}^m \to \mathbb{P}^n(\mathbb{F})$ be a linearly non-degenerate non-Archimedean*

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meromorphic map with index of independence s *and* $\operatorname{rank} f = k$. *Let* $H_1, \dots, H_q$ *be hyperplanes in* $\mathbb{P}^n(\mathbb{F})$ *in* N*-subgeneral position* $(N \geq n)$. *Then, for all* $r \geq 1$,

$$(q - 2N + n - 1)T_f(r) \leq \sum_{i=1}^{q} N_f^{(a)}(H_i, r) - \frac{N+1}{n+1}\log r + O(1),$$

where

$$a = \begin{cases} p^{s-1}(n-k+1) & \text{if } p > 0, \\ n-k+1 & \text{if } p = 0. \end{cases}$$

Here, the index of independence s and the $\operatorname{rank} f$ are defined in Section 2 (Definition 2.1).

Also, in 2017, An and Quang [2] proved a truncated second main theorem for meromorphic mappings from $\mathbb{C}^m$ into a projective variety $V \subset \mathbb{P}^M(\mathbb{C})$ with hypersurfaces. Motivated by the methods of Yan [6] and An-Quang [2], our aim in this article is to generalize Theorem A to the case where the map f is from $\mathbb{F}^m$ into an arbitrary projective variety V of dimension n in $\mathbb{P}^M(\mathbb{F})$ and the hyperplanes are replaced by hypersurfaces of $\mathbb{P}^M(\mathbb{F})$ in N-subgeneral position with respect to V.

Firstly, we give the following definitions.

Definition B. *Let* V *be a projective subvariety of* $\mathbb{P}^M(\mathbb{F})$ *of dimension* n $(n \leq M)$. *Let* $Q_1, \dots, Q_q$ $(q \geq n+1)$ *be* q *hypersurfaces in* $\mathbb{P}^M(\mathbb{F})$. *The family of hypersurfaces* $\{Q_i\}_{i=1}^q$ *is said to be in* N*-subgeneral position with respect to* V *if*

$$V \cap (\bigcap_{j=1}^{N+1} Q_{i_j}) = \varnothing \quad \text{for any } 1 \leq i_1 < \cdots < i_{N+1} \leq q.$$

If $N = n$, *we just say* $\{Q_i\}_{i=1}^q$ *is in general position with respect to* V.

Now, let V be as above and let d be a positive integer. We denote by $I(V)$ the ideal of homogeneous polynomials in $\mathbb{F}[x_0, \dots, x_M]$ defining V and by H_d the $\mathbb{F}$-vector space of all homogeneous polynomials in $\mathbb{F}[x_0, \dots, x_M]$ of degree d. Define

$$I_d(V) := \frac{H_d}{I(V) \cap H_d} \text{ and } H_V(d) := \dim_{\mathbb{F}} I_d(V).$$

Then $H_V(d)$ is called the Hilbert function of V. Each element of $I_d(V)$ which is an equivalent class of an element $Q \in H_d$, will be denoted by $[Q]$,

Definition C. *Let* $f : \mathbb{F}^m \to V$ *be a non-Archimedean meromorphic map with a reduced representation* $\mathbf{f} = (f_0, \dots, f_M)$. *We say that* f *is degenerate over* $I_d(V)$ *if there is* $[Q] \in I_d(V) \setminus \{0\}$ *such that* $Q(\mathbf{f}) \equiv 0$. *Otherwise, we say that* f *is non-degenerate over* $I_d(V)$.

We will generalize Theorem A to the following.

Theorem 1.1. *Let V be a projective subvariety of $\mathbb{P}^M(\mathbb{F})$ of dimension n ($n \le M$). Let $\{Q_i\}_{i=1}^q$ be hypersurfaces of $\mathbb{P}^M(\mathbb{F})$ in N-subgeneral position with respect to V with $\deg Q_i = d_i$ $(1 \le i \le q)$. Let d be the least common multiple of d_i's. Let f be a non-Archimedean meromorphic map of $\mathbb{F}^m$ into V, which is non-degenerate over $I_d(V)$ with the d^{th}-index of non-degeneracy s and $\mathrm{rank} f = k$. Then, for all $r \ge 1$,*

$$\left(q - \frac{(2N+n-1)H_d(V)}{n+1}\right) T_f(r) \le \sum_{i=1}^{q} \frac{1}{d_i} N_f^{(\kappa_0)}(Q_i, r) - \frac{N(H_d(V)-1)}{nd} \log r + O(1),$$

where

$$\kappa_0 = \begin{cases} p^{s-1}(H_d(V) - k) & \text{if } p > 0, \\ H_d(V) - k & \text{if } p = 0. \end{cases}$$

Here, the d^{th}-index of non-degeneracy s is defined in Section 2 (Definition 2.1). Note that, in the case where $V = \mathbb{P}^n(\mathbb{C}), d = 1, H_d(V) = n+1$, our result will give back Theorem A.

For the case of counting function without truncation level, we will prove the following.

Theorem 1.2. *Let V be a arbitrary projective subvariety of $\mathbb{P}^M(\mathbb{F})$. Let $\{Q_i\}_{i=1}^q$ be hypersurfaces of $\mathbb{P}^M(\mathbb{F})$ in N-subgeneral position with respect to V. Let f be a non-constant non-Archimedean meromorphic map of $\mathbb{F}^m$ into V. Then, for any $r > 0$,*

$$(q - N) T_f(r) \le \sum_{i=1}^{q} \frac{1}{\deg Q_i} N_f(Q_i, r) + O(1),$$

where the quantity $O(1)$ depends only on $\{Q_i\}_{i=1}^q$.

We see that, the above result is a generalization of the previous results in [1, 5].

2. Basic notions and auxiliary results

In this section, we will recall some basic notions from Nevanlinna theory for non-Archimedean meromorphic maps due to Cherry-Ye [3] and Yan [6].

2.1. Non-Archimedean meromorphic function. Let $\mathbb{F}$ be an algebraically closed field of characteristic p, complete with respect to a non-Archimedean absolute value $|\ |$. We set $\|z\| = \max_{1\le i\le m}|z_i|$ for $z = (z_1,\dots,z_m)\in\mathbb{F}^m$ and define

$$\mathbb{B}^m(r) := \{z\in\mathbb{F}^m; \|z\| < r\}.$$

For a multi-index $\gamma = (\gamma_1,\dots,\gamma_m)\in\mathbb{Z}^m_{\ge 0}$, define

$$z^\gamma = z_1^{\gamma_1}\cdots z_m^{\gamma_m},\ |\gamma| = \gamma_1+\cdots+\gamma_m,\ \gamma! = \gamma!\cdots\gamma_m!.$$

For an analytic function f on $\mathbb{F}^m$ (i.e., entire function) given by a formal power series

$$f = \sum_\gamma a_\gamma z^\gamma$$

with $a_\gamma\in\mathbb{F}$ such that $\lim_{|\gamma|\to\infty}|a_\gamma|r^{|\gamma|} = 0$ ($\forall r\in\mathbb{F}^* = \mathbb{F}\setminus\{0\}$), define

$$|f|_r = \sup_\gamma |a_\gamma|r^{|\gamma|}.$$

We denote by $\mathcal{E}_m$ the ring of all analytic functions on $\mathbb{F}^m$.

We define a meromorphic function f on $\mathbb{F}^m$ to be the quotient of two analytic functions $g,h\in\mathcal{E}_m$ such that g and h have no common factors in $\mathcal{E}_m$, i.e., $f = \frac{g}{h}$. We define

$$|f|_r = \frac{|g|_r}{|h|_r}.$$

We denote by $\mathcal{M}_m$ the field of all meromorphic functions on $\mathbb{F}^m$, which is the fractional field of $\mathcal{E}_m$.

2.2. Derivatives and Hasse derivatives. For a meromorphic function $f\in\mathcal{M}_m$ and a multi-index $\gamma = (\gamma_1,\dots,\gamma_m)$, we set

$$\partial^\gamma f = \frac{\partial^{|\gamma|}f}{\partial z_1^{\gamma_1}\cdots\partial z_m^{\gamma_m}}.$$

Let $\alpha = (\alpha_1,\dots,\alpha_m)$ and $\beta = (\beta_1,\dots,\beta_m)$ be multi-indices. We say that $\alpha\ge\beta$ if $\alpha_i\ge\beta_i$ for all $i = 1,\dots,m$. If $\alpha\ge\beta$, we define

$$\alpha-\beta = (\alpha_1-\beta_1,\dots,\alpha_m-\beta_m), \binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_m}{\beta_m}.$$

For an analytic function $f = \sum_\alpha a_\alpha z^\alpha$ and a multi-index γ, we define the Hasse derivative of multi-index γ of f by

$$D^\gamma f = \sum_{\alpha\ge\gamma}\binom{\alpha}{\gamma}a_\alpha z^{\alpha-\gamma}.$$

We may verify that $D^\alpha D^\beta f = \binom{\alpha+\beta}{\beta}D^{\alpha+\beta}$ for all $f\in\mathcal{E}_m$. Therefore, the Hasse derivative D can be extended to meromorphic functions in the following way:

- For a multi-index $e_i = (0, \ldots, 0, \underset{j^{th}-position}{1}, 0, \ldots, 0)$, we set $D_j^k f := D^{ke_i}(f)$.
- For a meromorphic function $f = \frac{g}{h}$ $(g, h \in \mathcal{E}_m)$, we define

$$D^{e_i} = D_j^1 f := \frac{hD_i^1 g - gD_i^1 h}{h^2},\ j = 1, \ldots, m.$$

- For $\gamma = (\gamma_1, \ldots, \gamma_m)$, we may choose a sequence of multi-indices $\gamma = \alpha^1 > \alpha^2 > \cdots > \alpha^{|\gamma|}$ such that $\alpha^i = \alpha^{i+1} + e_{j_i}$ $(j_i \in \{1, \ldots, m\})$ for $1 \leq i \leq |\gamma| - 1$ and $\alpha^{|\gamma|} = e_{j_{|\gamma|}}$ $(j_{|\gamma|} \in \{1, \ldots, m\})$ and define

$$D^{\alpha_i} h = \frac{1}{\binom{\alpha_{i+1}+e_{j_i}}{\alpha_{i+1}}} D^{e_{j_i}} D^{\alpha_{i+1}} h, \forall i = |\gamma| - 1, |\gamma| - 2, \ldots, 1.$$

We summarize here the fundamental properties of the Hasse derivative from [6] as follows:

(i) $D^\gamma(f+g) = D^\gamma f + D^\gamma g,\ f, g \in \mathcal{M}_m$.

(ii) $D^\gamma(fg) = \sum_{\alpha,\beta} D^\alpha f D^\beta g,\ f, g \in \mathcal{M}_m$.

(iii) $D^\alpha D^\beta f = \binom{\alpha+\beta}{\beta} D^{\alpha+\beta} f,\ f \in \mathcal{M}_m$

(iv) (Lemma on the logarithmic derivative) For $f \in \mathcal{E}_m$,

$$|D^\gamma f|_r \leq \frac{|f|_r}{r^{|\gamma|}},\ |\partial^\gamma f|_r \leq \frac{|f|_r}{r^{|\gamma|}}.$$

(v) For $f \in \mathcal{E}_m$ and a multi-index γ, let P be an irreducible element of $\mathcal{E}_m$ that divides f with exact multiplicity e. If $e > |\gamma|$, then $P^{e-|\gamma|}$ divides $D^\gamma f$.

For each integer $k \geq 2$, let

$$\mathcal{M}_m[k] = \{Q \in \mathcal{M}_m : D_j^i Q \equiv 0 \text{ for all } 0 < i < k \text{ and } 1 \leq j \leq m\}.$$

If F has characteristic 0, then $\mathcal{M}_m[k] = \mathbb{F}$ for all $k \geq 2$. If $\mathbb{F}$ has characteristic $p > 0$ and if $s \geq 1$ is an integer, then $\mathcal{M}_m[p^s]$ is the fraction field of $\mathcal{E}_m$, where $\mathcal{E}_m[p^s] = \{g^{p^s} : g \in \mathcal{E}_m\}$ is a subring of $\mathcal{E}_m$. Moreover,

$$\mathcal{M}_m[p^{s-1} + 1] = \mathcal{M}_m[p^s].$$

2.3. Non-Archimedean Nevanlinna's function.

Let $f = \sum_\gamma a_\gamma z^\gamma \in \mathcal{E}_m$ be an holomorphic function. The counting function of zeros of f is defined as follows:

$$N_f(0, r) = n_f(0, 0) \log r + \int_0^r (n_f(0, t) - n_f(0, 0)) \frac{dt}{t}\ (r > 0),$$

where

$$n_f(0,r) = \sup\{|\gamma|; |a_\gamma| r^{|\gamma|} = |f|_r\} \text{ and } n_f(0,0) = \min\{|\gamma|; a_\gamma \neq 0\}.$$

Let f be a meromorphic function on $\mathbb{F}^m$. Assume that $f = \frac{g}{h}$, where g, h are holomorphic functions without common factors. We define

$$N_f(0,r) = N_g(0,r) \text{ and } N_f(\infty,r) = N_h(0,r).$$

The Poisson-Jensen-Green formula (see [3, Theorem 3.1]) states that

$$N_f(0,r) - N_f(\infty,r) = \log|f|_r + C_f \quad \text{for all } r > 0,$$

where C_f is a constant depending on f but not on r.

Suppose that $f \not\equiv a$ for $a \in \mathbb{F}$. The counting function of f with respect to the point a is defined by

$$N_f(a,r) = N_{f-a}(0,r).$$

The proximity functions of f with respect to ∞ and a are defined respectively as follows

$$m_f(\infty,r) = \max\{0, \log|f|_r\} = \log^+|f|_r \text{ and } m_f(a,r) = m_{1/(f-a)}(\infty,r).$$

The characteristic function of f is defined by

$$T_f(r) = m_f(\infty,r) + N_f(\infty,r).$$

Note that, if $f = \frac{g}{h}$ as above then $T_f(r) = \max\{\log|g|_r, \log|h|_r\} + O(1)$.

The first main theorem is stated as follows:

$$T_f(r) = m_f(a,r) + N_f(a,r) + O(1) \ (\forall r > 0).$$

2.4. Truncated counting function.

Let $f \in \mathcal{E}_m$. For $j = 1, \ldots, m$, define

$$g_j = \gcd(f, D_j^1(f)) \text{ and } h_j = \frac{f}{g_j}.$$

The radical $R(f)$ of f is defined to be the least common multiple of h_j's.

Case 1: $\mathbb{F}$ has characteristic $p = 0$. The truncated counting function of zeros of f is defined by

$$N_f^{(l)}(0,r) = N_{\gcd(f,R(f)^l)}(0,r).$$

In particular,

$$N_f^{(1)}(0,r) = N_{R(f)}(0,r).$$

Case 2: $\mathbb{F}$ has characteristic $p > 0$. We define $R_{p^s}(f)$ by induction in $s = 0, 1, \ldots$ For $s = 0$, set $R_{p^0}(f) = R(f)$. For $s \geq 1$, assume that $R_{p^{s-1}}(f)$ has been defined. We set

$$\overline{f} = \frac{f}{\gcd(f, R_{p^{s-1}}(f)^{p^s})}, \; g_i = \gcd(\overline{f}, D_i^{p^s}\overline{f}), \; h_i = \frac{\overline{f}}{g_i}$$

for $i = 1, \ldots, m$. Let H be the least common multiple of h_i's, and set

$$G = \frac{H}{\gcd(H, R_{p^{s-1}}(H)^{p^{s-1}})},$$

which is a p^sth power. Let R be the p^sth root of G and define the higher p^s-radical $R_{p^s}(f)$ of f to be the least common multiple of $R_{p^{s-1}}(f)$ and R.

Take a sequence $\{r_j\}_{i\in\mathbb{N}} \subset |\mathbb{F}^*|$ such that $r_j \to \infty$. Take s_j such that if $P \in \mathcal{E}_m$ is irreducible such that $P|f$ and P is not unit on $\mathbb{B}^m(r_j)$ then $P|R_{p^s}(f)$ for $s > s_j$. Let u_j be a unit on $\mathbb{B}^m(r_j)$ such that

$$R_{p^{s_j}}(f) = u_j R_{p^{s_{j+1}}}(f).$$

Define $v_j = \prod_{l=j}^{\infty} u_j$, which is unit on $\mathbb{B}^m(r_j)$, and

$$S(f) = \lim_{j\to\infty} \frac{R_{p^{s_j}}(f)}{v_j} \in \mathcal{E}_m,$$

which is called the square free part of f. The truncated (to level l) counting function of zeros of f is defined by

$$N_f^{(l)}(0,r) = N_{\gcd(f,S(f)^l)}(0,r).$$

2.5. Non-Archimedean meromorphic maps and family of hypersurfaces.

Let V be a projective subvariety of $\mathbb{P}^M(\mathbb{F})$ of dimension n ($n \leq M$). For a positive integer d, take a basis $\{[A_1], \ldots, [A_{H_d(V)}]\}$ of $I_d(V)$, where $A_i \in \mathcal{H}_d[x_0, \ldots, x_M]$. Let $f : \mathbb{F}^m \to \mathbb{P}^M(\mathbb{F})$ be a non-Archimedean meromorphic map with a reduced representation $\mathbf{f} = (f_0, \ldots, f_M)$, which is non-degenerate over $I_d(V)$. We have the following definition.

Definition 2.1. *Assume that $\mathbb{F}$ has the character $p > 0$. Denote by s the smallest integer such that any subset of $\{A_1(\mathbf{f}), \ldots, A_{H_d(V)}(\mathbf{f})\}$ linearly independent over $\mathbb{F}$ remains linearly independent over $\mathcal{M}_m[p^s]$. We call s is the d^{th}-index of non-degeneracy of f.*

We see that the above definition does not depend on the choice of the basis $\{[A_i]; 1 \le i \le H_d(V)\}$ and the choice of the reduced representation $\mathbf{f}$. If $V = \mathbb{P}^M(\mathbb{F})$ and $d = 1$ then s is also called the index of independence of f (see [6, Definition 4.1]).

The following three lemmas are proved in [2] for the case of $\mathbb{F} = \mathbb{C}$ and the canonical absolute value. However, with the same proof, they also hold for arbitrary algebraic closed field $\mathbb{F}$ of character $p \ge 0$ and complete with an arbitrary absolute value. We state them here without the proofs.

Throughout this paper, we sometimes identify each hypersurface in a projective variety with its defining homogeneous polynomial. The following lemma of An-Quang [2] may be considered as a generalization of the lemma on Nochka weights in [4].

Lemma 2.1 (cf. [2, Lemma 3]). *Let V be a projective subvariety of $\mathbb{P}^M(\mathbb{F})$ of dimension n $(n \le M)$. Let $Q_1, \dots, Q_q$ be q $(q > 2N - k + 1)$ hypersurfaces in $\mathbb{P}^M(\mathbb{F})$ in N-subgeneral position with respect to V of the common degree d. Then there are positive rational constants ω_i $(1 \le i \le q)$ satisfying the following:*

i) $0 < \omega_i \le 1,\ \forall i \in \{1, \dots, q\}$,

ii) Setting $\tilde{\omega} = \max_{j \in Q} \omega_j$, *one gets*

$$\sum_{j=1}^{q} \omega_j = \tilde{\omega}(q - 2N + n - 1) + n + 1.$$

iii) $\dfrac{n+1}{2N - n + 1} \le \tilde{\omega} \le \dfrac{n}{N}$.

iv) For $R \subset \{1, \dots, q\}$ *with* $\sharp R = N + 1$, *then* $\sum_{i \in R} \omega_i \le n + 1$.

v) Let $E_i \ge 1$ $(1 \le i \le q)$ *be arbitrarily given numbers. For* $R \subset \{1, \dots, q\}$ *with* $\sharp R = N + 1$, *there is a subset* $R^o \subset R$ *such that* $\sharp R^o = \mathrm{rank}_{\mathbb{F}}\{[Q_i]; i \in R^o\} = n + 1$ *and*

$$\prod_{i \in R} E_i^{\omega_i} \le \prod_{i \in R^o} E_i.$$

Let Q be a hypersurface in $\mathbb{P}^n(\mathbb{F})$ of degree d defined by $\sum_{I \in \mathcal{I}_d} a_I x^I = 0$, where $\mathcal{I}_d = \{(i_0, \dots, i_M) \in \mathbb{N}_0^{M+1}\ :\ i_0 + \cdots + i_M = d\}$, $I = (i_0, \dots, i_M) \in \mathcal{I}_d$, $x^I = x_0^{i_0} \cdots x_M^{i_M}$ and $(x_0 : \cdots : x_M)$ is homogeneous coordinates of $\mathbb{P}^M(\mathbb{F})$. Let f be an non-Archimedean meromorphic map from $\mathbb{F}^m$ into a projective subvariety V of $\mathbb{P}^M(\mathbb{F})$ with a reduced representation $\mathbf{f} = (f_0, \dots, f_M)$. We define

$$Q(\mathbf{f}) = \sum_{I \in \mathcal{I}_d} a_I f^I,$$

where $f^I = f_0^{i_0} \cdots f_n^{i_n}$ for $I = (i_0, \dots, i_n)$. We have the following lemma.

Lemma 2.2 (cf. [2, Lemma 4]). *Let $\{Q_i\}_{i\in R}$ be a set of hypersurfaces in $\mathbb{P}^n(\mathbb{F})$ of the common degree d and let f be a meromorphic mapping of $\mathbb{F}^m$ into $\mathbb{P}^n(\mathbb{F})$ with a reduced representation $\mathbf{f}=(f_0,\dots,f_M)$. Assume that $\bigcap_{i\in R} Q_i\cap V=\varnothing$. Then, there exist positive constants α and β such that*

$$\alpha\|\mathbf{f}\|_r^d\le \max_{i\in R}|Q_i(\mathbf{f})|_r\le\beta\|\mathbf{f}\|_r^d \text{ for any } r>0.$$

Lemma 2.3 (cf. [2, Lemma 5]). *Let $\{Q_i\}_{i=1}^q$ be a set of q hypersurfaces in $\mathbb{P}^M(\mathbb{F})$ of the common degree d. Then there exist $(H_V(d)-n-1)$ hypersurfaces $\{T_i\}_{i=1}^{H_V(d)-n-1}$ in $\mathbb{P}^M(\mathbb{F})$ such that for any subset $R\in\{1,\dots,q\}$ with $\sharp R=\operatorname{rank}_{\mathbb{F}}\{[Q_i];i\in R\}=n+1$, we get $\operatorname{rank}_{\mathbb{F}}\{\{[Q_i];i\in R\}\cup\{[T_i];1\le i\le H_d(V)-n-1\}\}=H_V(d)$.*

2.5. Value distribution theory for non-Archimedean meromorphic maps.

Let $f:\mathbb{F}^m\to V\subset\mathbb{P}^M(\mathbb{F})$ be a non-Archimedean meromorphic map with a reduced representation $\mathbf{f}=(f_0,\dots,f_N)$. The characteristic function of f is defined by

$$T_f(r)=\log\|\mathbf{f}\|_r,$$

where $\|\mathbf{f}\|_r=\max_{1\le 0\le n}|f_i|_r$. This definition is well-defined upto a constant.

Let Q be a hypersurface in $\mathbb{P}^n(\mathbb{F})$ of degree d defined by $\sum_{I\in\mathcal{I}_d}a_Ix^I=0$, where $a_I\in\mathbb{F}$ $(I\in\mathcal{I}_d)$ and are not all zeros. If $Q(\mathbf{f})\not\equiv 0$ then we define the proximity function of f with respect to Q by

$$m_f(Q,r)=\log\frac{\|\mathbf{f}\|_r^d\cdot\|Q\|}{|Q(\mathbf{f})|_r},$$

where $\|Q\|:=\max_{I\in\mathcal{I}_d}|a_I|$. We see that the definition of $m_f(Q,r)$ does not depend on the choices of the presentations of f and Q.

The truncated (to level l) counting function of f with respect to Q is defined by

$$N_f^{(l)}(Q,r):=N_{Q(\mathbf{f})}^{(l)}(0,r).$$

For simplicity, we will omit the character $^{(l)}$ if $l=\infty$.

The first main theorem for non-Archimedean meromorphic maps states that

$$dT_f(r)=m_f(Q,r)+N_f(Q,r)+O(1).$$

Proposition 2.1 (cf. [6, Propositions 4.3, 4.4]). *Let p be the character of $\mathbb{F}$. Assume that $f:\mathbb{F}_m\to\mathbb{P}^n(\mathbb{F})$ is a non-Achimedean meromorphic map, which is* *linearly non-degenerate over* $\mathbb{F}$*, with a reduced representation* $\mathbf{f} = (f_0, \ldots, f_n)$*. Then there exist multi-indices* $\gamma^0 = (0, \ldots, 0), \gamma^1, \ldots, \gamma^n$ *with*

$$|\gamma^0| \leq \cdots \leq |\gamma^n| \leq \kappa_0 \leq \begin{cases} p^{s-1}(n-k+1) & \text{if } p > 0, \\ n-k+1 & \text{if } p = 0 \end{cases}$$

where s *is the index of independence of* f *and* $k = \operatorname{rank} f$*, such that the generalized Wronskian*

$$W_{\gamma^0,\ldots,\gamma^n}(f_0, \ldots, f_n) = \det\left(D^{\gamma^i} f_j\right)_{0 \leq i,j \leq n} \not\equiv 0.$$

Here $\operatorname{rank} f$ is defined by

$$\operatorname{rank} f = \operatorname{rank}_{\mathcal{M}_m}\{(D^\gamma f_0, \ldots, D^\gamma f_n); |\gamma| \leq 1\} - 1.$$

3. Proof of main theorems

Proof. [Proof of Theorem 1.1] By replacing Q_i with Q_i^{d/d_i} if necessary, we may assume that all Q_i $(i = 1, \ldots, q)$ do have the same degree d. It is easy to see that there is a positive constant β such that $\beta\|\mathbf{f}\|^d \geq |Q_i(\mathbf{f})|$ for every $1 \leq i \leq q$. Set $Q := \{1, \cdots, q\}$. Let $\{\omega_i\}_{i=1}^q$ be as in Lemma 2.1 for the family $\{Q_i\}_{i=1}^q$. Let $\{T_i\}_{i=1}^{H_d(V)-n-1}$ be $(H_d(V)-n-1)$ hypersurfaces in $\mathbb{P}^M(\mathbb{F})$, which satisfy Lemma 2.3.

Take a $\mathbb{F}$-basis $\{[A_i]\}_{i=1}^{H_V(d)}$ of $I_d(V)$, where $A_i \in H_d$. Since f is non-degenerate over $I_d(V)$, it implies that $\{A_i(\mathbf{f}); 1 \leq i \leq H_V(d)\}$ is linearly independent over $\mathbb{F}$. By Proposition 2.1, there multi-indices $\{\gamma^1 = (0, \ldots, 0), \gamma^2 \cdots, \gamma^{H_V(d)}\} \subset \mathbb{Z}_+^m$ such that $|\gamma^0| \leq \cdots \leq |\gamma^{H_d(V)}| \leq \kappa_0$, where

$$\kappa_0 \leq \begin{cases} p^{s-1}(H_V(d) - k) & \text{if } p > 0, \\ H_d(V) - k & \text{if } p = 0 \end{cases}$$

and the generalized Wronskian

$$W = \det\left(D^{\gamma^i} A_j(\mathbf{f})\right)_{1 \leq i,j \leq H_d(V)} \not\equiv 0.$$

Here, we note that

$$\begin{aligned} k &= \operatorname{rank}_{\mathcal{M}_m}\{(D^\gamma f_0,\dots,D^\gamma f_M);|\gamma|\le 1\}-1 \\ &= \operatorname{rank}_{\mathcal{M}_m}\left\{\left(D^\gamma(\frac{f_1}{f_0}),\dots,D^\gamma(\frac{f_M}{f_0})\right);|\gamma|\le 1\right\} \\ &\le \operatorname{rank}_{\mathcal{M}_m}\left\{\left(D^\gamma(\frac{A_2(\mathbf{f})}{A_1(\mathbf{f})}),\dots,D^\gamma(\frac{A_{H_d(V)}(\mathbf{f})}{A_1(\mathbf{f})})\right);|\gamma|\le 1\right\} \\ &= \operatorname{rank}_{\mathcal{M}_m}\left\{(D^\gamma(A_1(\mathbf{f})),\dots,D^\gamma(A_{H_d(V)}(\mathbf{f})));|\gamma|\le 1\right\}-1. \end{aligned}$$

For each $R^o=\{r_1^0,\dots,r_{n+1}^0\}\subset\{1,\dots,q\}$ with $\operatorname{rank}_{\mathbb{F}}\{Q_i\}_{i\in R^o}=\sharp R^o=n+1$, set

$$W_{R^o}\equiv\det\big(D^{\gamma^j}Q_{r_v^0}(\mathbf{f})(1\le v\le n{+}1),D^{\gamma^j}T_l(\mathbf{f})(1\le l\le H_V(d){-}n{-}1)\big)_{1\le j\le H_V(d)}.$$

Since $\operatorname{rank}_{\mathbb{F}}\{[Q_{r_v^0}](1\le v\le n+1),[T_l](1\le l\le H_V(d)-n-1)\}=H_V(d)$, there exists a nonzero constant $C_{R^o}\in\mathbb{F}$ such that $W_{R^o}=C_{R^o}\cdot W$.

We denote by $\mathcal{R}^o$ the family of all subsets R^o of $\{1,\dots,q\}$ satisfying

$$\operatorname{rank}_{\mathbb{F}}\{[Q_i];i\in R^o\}=\sharp R^o=n+1.$$

For each $r>0$, there exists $\bar R\subset Q$ with $\sharp\bar R=N+1$ such that $|Q_i(\mathbf{f})|_r\le|Q_j(\mathbf{f})|_r,\forall i\in\bar R,j\notin\bar R$. We choose $R^o\subset R$ such that $R^o\in\mathcal{R}^o$ and R^o satisfies Lemma 2.1(v) with respect to numbers $\{\dfrac{\beta\|\mathbf{f}\|_r^d}{|Q_i(\mathbf{f})|_r}\}_{i=1}^q$. Since $\bigcap_{i\in\bar R}Q_i=\varnothing$, by Lemma 2.2, there exists a positive constant $\alpha^{\bar R}$ such that

$$\alpha^{\bar R}\|\mathbf{f}\|_r^d\le\max_{i\in\bar R}|Q_i(\mathbf{f})|_r.$$

Then, we get

$$\begin{aligned} \frac{\|\mathbf{f}\|_r^{d(\sum_{i=1}^q\omega_i)}|W|_r}{|Q_1(\mathbf{f})|_r^{\omega_1}\cdots|Q_q(\mathbf{f})|_r^{\omega_q}} &\le \frac{|W|_r}{\alpha_{\bar R}^{q-N-1}\beta^{N+1}}\prod_{i\in\bar R}\left(\frac{\beta\|\mathbf{f}\|_r^d}{|Q_i(\mathbf{f})|_r}\right)^{\omega_i} \\ &\le A_{\bar R}\frac{|W|_r\cdot\|\mathbf{f}\|_r^{d(n+1)}}{\prod_{i\in\bar R^o}|Q_i(\mathbf{f})|_r} \\ &\le B_{\bar R}\frac{|W_{\bar R^o}|_r\cdot\|\mathbf{f}\|_r^{dH_V(d)}}{\prod_{i\in\bar R^o}|Q_i(\mathbf{f})|_r\prod_{i=1}^{H_V(d)-n-1}|T_i(\mathbf{f})|_r}, \end{aligned}$$

where $A_{\bar R},B_{\bar R}$ are positive constants.

Therefore, for every $r > 0$,

$$\log \frac{\|\mathbf{f}\|_r^{d(\sum_{i=1}^q \omega_i - H_d(V)}|W|_r}{|Q_1(\mathbf{f})|_r^{\omega_1}\cdots|Q_q(\mathbf{f})|_r^{\omega_q}} \le \max_R \log \frac{|W_R|_r}{\prod_{i\in R}|Q_i(\mathbf{f})|_r \prod_{i=1}^{H_V(d)-n-1}|T_i(\mathbf{f})|_r} + O(1)$$
$$\le -\sum_{j=1}^{H_d(V)} |\gamma^j| \log r + O(1),$$

where the maximum is taken over all subsets $R \subset \{1,\dots,q\}$ such that $\sharp R = n+1$ and $\mathrm{rank}_{\mathbb{F}}\{[Q_i]; i \in R\} = n+1$. Here, the last inequality comes from the lemma on logarithmic derivative. By the Poisson-Jensen-Green formula, the definitions of the approximation function and the characteristic function, we have

$$\sum_{i=1}^q \omega_i m_f(Q_i, r) - dH_d(V)T_f(r) - N_W(0,r) \le -(H_d(V)-1)\log r + O(1),$$

(note that $\sum_{i=1}^{H_d(V)} |\gamma^i| \le H_d(V) - 1$). Then, by the first main theorem, we obtain

(3.1)
$$(\sum_{i=1}^q \omega_i - H_d(V))dT_f(r) \le \sum_{i=1}^q \omega_i N_f(Q_i, r) - N_W(0,r) - (H_d(V)-1)\log r + O(1).$$

Claim. $\sum_{i=1}^q \omega_i N_f(Q_i, r) - N_W(0,r) \le \sum_{i=1}^q \omega_i N_f^{(\kappa_0)}(Q_i, r) + O(1)$.

Indeed, set $\tilde{G}_j = \gcd(Q_j(\mathbf{f}), S(Q_j(\mathbf{f}))^{\kappa_0})$. Since ω_i $(1 \le i \le q)$ are rational numbers, there exists an integer A such that $\tilde{\omega}_i = A\omega_i$ $(1 \le i \le q)$ are integers.

Let $P \in \mathcal{E}_m$ be an irreducible element with $P|\prod_{i=1}^q Q_i(\mathbf{f})^{\tilde{\omega}_i}$. There exists a subset R of $\{1,\dots,q\}$ with $\sharp R = N+1$ such that P is not a division of $Q_i(\mathbf{f})$ for any $i \notin R$. Denote by e_i the largest integer such that $P^{e_i}|Q_i(\mathbf{f})$ for each $i \in R$. Then, there is a subset $R^o \subset R$ with $\sharp R^o = n+1$, $W_{R^o} \not\equiv 0$ and

$$\sum_{i\in R} \omega_i \max\{0, e_i - \kappa_0\} \le \sum_{i \in R^o} \max\{0, e_i - \kappa_0\}.$$

Also, since $W = C_{R^o}\cdot W_{R^o}$, it clear that P divides W with multiplicity at least

$$\min_{\{j_1,\dots,j_{n+1}\}\subset\{1,\dots,H_d(V)\}} \sum_{i\in R^0} \min\{0, e_i - |\gamma^{j_i}|\} \ge \sum_{i\in R^0} \min\{0, e_i - \kappa_0\}$$
$$\ge \sum_{i\in R} \omega_i \max\{0, e_i - \kappa_0\}$$
$$= \sum_{i\in R} \omega_i (e_i - \min\{e_i, \kappa_0\}).$$

This implies that

$$P^{\sum_{i\in R}\tilde{\omega}_i e_i} | W^A \cdot P^{\sum_{i\in R}\tilde{\omega}_i \min\{e_i,\kappa_0\}}.$$

We note that $P^{\tilde{\omega}_i \min\{e_i,\kappa_0\}} | G_i^{\tilde{\omega}_i}$. Therefore,

$$P^{\sum_{i\in R}\tilde{\omega}_i e_i} | W^A \cdot \prod_{i\in R} G_i^{\tilde{\omega}_i}.$$

This holds for every such irreducible element P. Then it yields that

$$\prod_{i=1}^{q} Q_i(\mathbf{f})^{\tilde{\omega}_i} | W^A \cdot \prod_{i=1}^{q} G_i^{\tilde{\omega}_i}.$$

Hence,

$$\sum_{i=1}^{q} N_f(Q_i,r) \leq N_W(0,r) + \sum_{i=1}^{q} N_f^{(\kappa_0)}(Q_i,r).$$

The claim is proved.

From the claim, Lemma 2.1(ii) and the inequality (3.1), we obtain

$$(\tilde{\omega}(q-2N+n-1) - H_d(V) + n + 1)dT_f(r) \leq \sum_{i=1}^{q} \omega_i N_f^{(\kappa_0)}(Q_i,r) - (H_d(V)-1)\log r + O(1).$$

Note that, $\omega_i \leq \tilde{\omega}(1 \leq i \leq q)$ and $\dfrac{n+1}{2N-n+1} \leq \tilde{\omega} \leq \dfrac{n}{N}$. Then, the above inequality implies that

$$\left(q - \frac{(2N-n+1)H_d(V)}{n+1}\right) \leq \sum_{i=1}^{q} \frac{1}{d} N_f^{(\kappa_0)}(Q_i,r) - \frac{N(H_d(V)-1)}{nd}\log r + O(1).$$

The theorem is proved.

Proof. [Proof of Theorem 1.2] For $r > 0$, without loss of generality, we may assume that

$$|Q_1(\mathbf{f})|_r^{1/\deg Q_1} \leq |Q_2(\mathbf{f})|_r^{1/\deg Q_2} \leq \cdots \leq |Q_q(\mathbf{f})|_r^{1/\deg Q_{N+1}}.$$

Since $\bigcap_{i=1}^{N+1} Q_i = \varnothing$, by Lemma 2.2, there exists a positive constant C such that

$$C\|\mathbf{f}\|_r \leq \max_{1\leq i\leq N+1} |Q_i(\mathbf{f})|_r^{1/\deg Q_i} = |Q_{N+1}(\mathbf{f})|_r^{1/\deg Q_{N+1}}.$$

Then, we get

$$\begin{aligned}
\sum_{i=1}^{q}\frac{m_f(Q_i,r)}{\deg Q_i} &= \log\frac{\|\mathbf{f}\|_r^q}{|Q_1(\mathbf{f})|_r^{1/\deg Q_1}\cdots|Q_q(\mathbf{f})|_r^{1/\deg Q_q}}+O(1)\\
&\le \log\prod_{i=1}^{N}\frac{\|\mathbf{f}\|_r}{|Q_i(\mathbf{f})|_r^{1/\deg Q_i}}+O(1)\\
&=\sum_{i=1}^{N}\frac{m_f(Q_i,r)}{\deg Q_i}+O(1)\\
&\le N\cdot T_f(r)+O(1).
\end{aligned}$$

Therefore,

$$(q-N)T_f(r)\le\sum_{i=1}^{q}\frac{1}{\deg Q_i}N_f(Q_i,r)+O(1)\quad (r>0).$$

The theorem is proved.

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